comments appreciated

Necessity of Transversality Conditions for Stochastic Problems

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Abstract

This paper establishes (i) an extension of Michel's (1990, Theorem 1) necessity result to an abstract reduced-form model, (ii) a generalization of the results of Weitzman (1973) and Ekeland and Scheinkman (1986), and (iii) a new result that is useful particularly in the case of homogeneous returns. These results are shown for an extremely general discrete-time reduced-form model that does not assume differentiability, continuity, or concavity, and that imposes virtually no restriction on the state spaces. The three results are further extended to a stochastic reduced-form model. The stochastic extensions are easily accomplished since our deterministic model is so general that the stochastic model is in fact a special case of the deterministic model. We apply our stochastic results to a stochastic reduced-form model with homogeneous returns and a general type of stochastic growth model with CRRA utility.

Keywords: Transversality condition, stochastic optimization, stochastic reduced-form model, homogeneous returns, stochastic growth model.

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1 Introduction

Since Shell (1969) and Halkin (1974), necessity of transversality conditions (TVCs) has been an uneasy matter to economic theorists who use infinite-horizon optimization problems. While various results on necessity of TVCs for deterministic continuous-time problems are established in Kamihigashi (2001a), the existing literature does not provide widely applicable results on necessity of TVCs for stochastic problems.

The most standard type of TVC, which we call the standard TVC (STVC), is the condition that the value of optimal stocks at infinity must be zero. The literature on necessity of the STVC dates back at least to Peleg (1970) and Weitzman (1973). For a deterministic concave problem, Weitzman shows that the STVC is necessary if the return functions are nonnegative and if the objective function is always finite.[1] Peleg shows the same result for a special case.[2] For a concave optimal control problem, Michel (1990) studies more general TVCs that an optimal path has to satisfy against a feasible path that does not cause an infinite loss. Michel's results, however, do not directly deal with the STVC.[3]

While the above results in fact provide characterizations of optimal paths for concave problems, Ekeland and Scheinkman (1986) focus on necessity of the STVC for a possibly non-concave problem. They show that the STVC is necessary if the utility functions satisfy a certain condition and if there exists a summable nonnegative sequence that majorizes the utility functions for all feasible paths.

All the aforementioned results, however, are concerned with deterministic problems. When it comes to stochastic problems, the knowledge on this issue is severely limited. Stochastic versions of Weitzman's theorem are shown by Zilcha (1976) and Takekuma (1992), but they require additional restrictive assumptions.[4] Stochastic versions of the results of Michel (1990) and Ekeland and Scheinkman (1986) are not available in the literature.

Among the main results of this paper are a stochastic version of the necessity part of Michel (1990, Theorem 1) and a stochastic version of a generalization of the TVC results of

[1] See Kamihigashi (2001a) for a simple proof of this result.
[2] See also Peleg and Ryder (1972) for a similar result.
[3] See Kamihigashi (2001a) for discussions of related results for continuous-time models.
[4] See Zilcha (1978) for results specific to an undiscounted stationary model.

Weitzman (1973) and Ekeland and Scheinkman (1986).

This paper extends the main results of Kamihigashi (2001a) to discrete-time problems. Both deterministic and stochastic cases are considered. Our model in the deterministic case is an extremely general reduced-form model. The model is general in that it does not assume differentiability, continuity, or concavity, and in that the state spaces are arbitrary vector spaces instead of finite-dimensional spaces. Because of this generality, our model in the stochastic case, which is a natural extension of the deterministic model, is in fact a special case of the deterministic model.[5]

For the deterministic case, we extend Michel's necessity result to our abstract reduced-form model. We also establish a result that simultaneously generalizes the TVC results of Weitzman (1973) and Ekeland and Scheinkman (1986). This generalization is significant especially because it does not require Ekeland and Scheinkman's assumption that there exists a summable nonnegative sequence that majorizes the utility functions for all feasible paths. In addition we obtain a new result that is useful particularly in the case of homogeneous returns. While similar results are shown in Kamihigashi (2001a) for deterministic continuous-time models, the results in this paper are shown in a substantially more general setting.[6] Furthermore our deterministic results are extended to the stochastic case. The stochastic extensions are easily accomplished since, as mentioned above, the stochastic model is a special case of the deterministic model.

We follow Ekeland and Scheinkman (1986) in using directional derivatives instead of support prices. The concavity-based results mentioned above use support prices and thus rely heavily on the separation theorem.[7] As is well-known, the separation theorem for infinite-dimensional spaces requires severe restrictions. This is one of the reasons why the results of Zilcha (1976) and Takekuma (1992) are not easily applicable. By contrast our results do not require such restrictions since we use directional derivatives instead of support prices. In fact

[5] The idea that a stochastic model can be viewed as a deterministic model is used by Yano (1989) to study the comparative statics of the stationary state of a stochastic growth model.

[6] While Kamihigashi (2001a) assumes that the return functions are differentiable, that the optimal path is interior, and that the state spaces are time-invariant and finite-dimensional, none of these assumptions is used in this paper.

[7] It should be mentioned that the difficult part in establishing these results is the construction of support prices, not the proof of the necessity of TVCs.

we use a generalized type of directional derivative that is well-defined for any real-valued function whose domain lies in a vector space. This allows us to concentrate on conditions directly related to TVCs.

Another feature of our approach is that multidimensional problems are transformed into one dimensional problems. Since a problem with any state spaces is reduced to a problem with a one-dimensional state space, our results are established in extremely general settings without introducing additional complexities.

The rest of the paper is organized as follows. Section 2 establishes our deterministic results. Section 3 extends them to the stochastic case. Section 4 applies our stochastic results to a stochastic reduced-form model with homogeneous returns and to a general type of stochastic growth model with CRRA utility. Section 5 concludes the paper. All the proofs are collected in Appendix A. Section A.1 establishes the most general versions of our results in a one-dimensional setting. The main results of the paper are proved by simple applications of the one-dimensional results.

2 The Deterministic Case

Consider the following problem.

$$\text{(2.1)} \qquad \begin{cases} \text{“}\max\limits_{\{x_t\}_{t=0}^{\infty}} & \sum\limits_{t=0}^{\infty} V_t(x_t, x_{t+1})\text{”} \\ \quad \text{s.t.} & x_0 = \overline{x}_0, \quad \forall t \in \mathbb{Z}_+, (x_t, x_{t+1}) \in X_t. \end{cases}$$

This section assumes the following assumptions.

Assumption 2.1. There exists a sequence of real vector spaces $\{D_t\}_{t=0}^{\infty}$ such that $\overline{x}_0 \in D_0$ and $\forall t \in \mathbb{Z}_+, X_t \subset D_t \times D_{t+1}$.

Assumption 2.2. $\forall t \in \mathbb{Z}_+, V_t : X_t \to [-\infty, \infty)$.

Assumptions 2.1 and 2.2 impose virtually no restriction on the model. Since each D_t is allowed to be an arbitrary vector space, it may even be a space of random variables. Therefore, though we consider (2.1) as a deterministic problem here, it in fact includes stochastic problems as special cases. We can work with this abstract setting since the only operations we need on D_t are addition and scalar multiplication.

We say that a sequence $\{x_t\}_{t=0}^{\infty}$ is a *feasible path* if $x_0 = \overline{x}_0$ and $\forall t \in \mathbb{Z}_+, (x_t, x_{t+1}) \in X_t$. Since in applications the objective function is often not guaranteed to be finite or well-defined for all feasible paths, we use weak maximality (Brock, 1970) as our optimality criterion. We say that a feasible path $\{x_t^*\}$ is *optimal* if for any feasible path $\{x_t\}$,

$$\varliminf_{T\uparrow\infty} \sum_{t=0}^{T} [V_t(x_t, x_{t+1}) - V_t(x_t^*, x_{t+1}^*)] \leq 0.\tag{2.2}$$ [8]

Remark 2.1. Suppose $\sum_{t=0}^{\infty} V_t(x_t, x_{t+1})$ exists in $[-\infty, \infty)$ for all feasible paths $\{x_t\}$.[9] Suppose $\sup \sum_{t=0}^{\infty} V_t(x_t, x_{t+1})$ is finite, where the sup is taken over all feasible paths $\{x_t\}$. Then a feasible path $\{x_t^*\}$ is optimal iff for any feasible path $\{x_t\}$,

$$\sum_{t=0}^{\infty} V_t(x_t, x_{t+1}) \leq \sum_{t=0}^{\infty} V_t(x_t^*, x_{t+1}^*).\tag{2.3}$$

Therefore, our optimality criterion coincides with the usual maximization criterion whenever the latter makes sense.

Our optimality criterion applies even when the usual maximization criterion fails. In addition, it is weaker than the similar criterion with $\varlimsup$ replacing $\varliminf$ in (2.2). Thus, though weak maximality is rarely directly used in applications, our results apply to virtually any discrete-time problem. The rest of this section assumes the following.

Assumption 2.3. There exists an optimal path $\{x_t^*\}$.

Since we are only interested in necessary conditions for optimality, this assumption imposes no restriction on the model.

For $t \in \mathbb{Z}_+$ and $d \in D_{t+1}$ such that $(x_t^*, x_{t+1}^* + \epsilon d) \in X_t$ for $\epsilon > 0$ sufficiently small, define

$$V_{t,2}(x_t^*, x_{t+1}^*; d) = \lim_{\epsilon\downarrow 0} \frac{V_t(x_t^*, x_{t+1}^* + \epsilon d) - V_t(x_t^*, x_{t+1}^*)}{\epsilon}.\tag{2.4}$$

[8]To be precise, this inequality requires that the left-hand side is well-defined. This means that the left-hand side does not involve expressions like "$\infty - \infty$" and "$-\infty + \infty$." An implication of this requirement is that $\forall t \in \mathbb{Z}_+, V_t(x_t^*, x_{t+1}^*)$ is finite; for otherwise the left-hand side of (2.2) is undefined for $\{x_t\} = \{x_t^*\}$.

[9]Throughout this paper, $\forall i \in \mathbb{Z}_+, \sum_{t=i}^{\infty} \equiv \lim_{T\uparrow\infty} \sum_{t=i}^{T}$. Such sums are not to be interpreted as Lebesgue integrals.

The right-hand side is always well-defined, though possibly equal to $-\infty$ or ∞, even if V_t is nondifferentiable or discontinuous. Note that if V_t is partially differentiable (in an appropriate sense) with respect to the second argument at (x_t^*, x_{t+1}^*) and if $V_{t,2}(x_t^*, x_{t+1}^*)$ denotes the partial derivative, then

$$V_{t,2}(x_t^*, x_{t+1}^*; d) = V_{t,2}(x_t^*, x_{t+1}^*)d. \tag{2.5}$$

Remark 2.2. Theorems 2.1–2.3 below hold even if $\overline{\lim}$ replaces $\underline{\lim}$ in (2.4).[10]

Theorem 2.1. *Assume Assumptions 2.1–2.3. Suppose* $\forall t \in \mathbb{Z}_+, X_t$ *is convex and* V_t *is concave. Then*

$$\underline{\lim}_{t\uparrow\infty} V_{t,2}(x_t^*, x_{t+1}^*; x_{t+1} - x_{t+1}^*) \leq 0 \tag{2.6}$$

for any feasible path $\{x_t\}$ *such that*

$$\underline{\lim}_{T\uparrow\infty} \sum_{t=0}^{T} [V_t(x_t, x_{t+1}) - V_t(x_t^*, x_{t+1}^*)] > -\infty, \tag{2.7}$$

$$\forall t \in \mathbb{Z}_+, \exists \epsilon > 0, \quad (x_t^*, x_{t+1}^* + \epsilon(x_{t+1} - x_{t+1}^*)) \in X_t. \tag{2.8}$$

Theorem 2.1 is a discrete-time version of Kamihigashi (2001a, Corollary 3.1); recall footnote 6. The necessity part of Michel (1990, Theorem 1) shows a similar result for a special case of (2.1) that assumes, among other things, finite-dimensional state spaces. He shows that there exists a sequence of support price vectors $\{p_t\}$ such that $\underline{\lim}_{t\uparrow\infty} p_t(x_t^* - x_t) \leq 0$ for any feasible path $\{x_t\}$ satisfying (2.7). In our case, $V_{t,2}(x_t^*, x_{t+1}^*; x_{t+1} - x_{t+1}^*)$ corresponds to $p_t(x_t^* - x_t)$. Condition (2.8) is needed here for $V_{t,2}(x_t^*, x_{t+1}^*; x_{t+1} - x_{t+1}^*)$ to be well-defined. Except for these differences, Theorem 2.1 generalizes Michel's necessity result to our abstract setting.

The rest of this section assumes the following.

Assumption 2.4. $\forall t \in \mathbb{Z}_+, \exists \underline{\lambda}_t \in [0,1), \forall \lambda \in [\underline{\lambda}_t, 1), (x_t^*, \lambda x_{t+1}^*) \in X_t$ and $\forall \tau \geq t + 1, (\lambda x_\tau^*, \lambda x_{\tau+1}^*) \in X_\tau$.

Remark 2.3. Assumption 2.4 holds if $\forall t \in \mathbb{Z}_+, X_t$ is convex and $(x_t^*, 0), (0,0) \in X_t$.

[10] See footnote 20 for why this remark is true. See (3.9) for why we use $\underline{\lim}$ instead of $\overline{\lim}$ in (2.4).

Assumption 2.4 means that the optimal path can be shifted proportionally downward starting from any period. The assumption is common to the well-known results on the STVC in the literature since the STVC basically means that no gain should be achieved by shifting the optimal path proportionally downward.

For $t \in \mathbb{N}$ and $\lambda \in \mathbb{R} \setminus \{1\}$ with $(\lambda x_t^*, \lambda x_{t+1}^*) \in X_t$, define

$$w_t(\lambda) = \frac{V_t(x_t^*, x_{t+1}^*) - V_t(\lambda x_t^*, \lambda x_{t+1}^*)}{1 - \lambda}, \tag{2.9}$$

$$\hat{w}_t(\lambda) = \sup_{\tilde{\lambda} \in [\lambda, 1)} w_t(\tilde{\lambda}), \tag{2.10}$$

where $\hat{w}_t(\lambda)$ is defined only for $\lambda \in [\underline{\lambda}_0, 1)$ ($\underline{\lambda}_0$ is given by Assumption 2.4).

Remark 2.4. For $t \in \mathbb{N}$, if $V_t(\lambda x_t^*, \lambda x_{t+1}^*)$ is concave in $\lambda \in [\underline{\lambda}_0, 1]$, then $\forall \lambda \in [\underline{\lambda}_0, 1), \hat{w}_t(\lambda) = w_t(\lambda)$.

Theorem 2.2. *Assume Assumptions 2.1–2.4. Suppose*

$$\exists \{b_t\}_{t=1}^{\infty} \subset \mathbb{R}, \exists \lambda \in [\underline{\lambda}_0, 1), \forall t \in \mathbb{N}, \qquad \hat{w}_t(\lambda) \leq b_t. \tag{2.11}$$

Then (i) *(2.12) ⇒ (2.13) and* (ii) *(2.14) ⇒ (2.15), where (2.12)–(2.15) are given by*

$$\overline{\lim_{T \uparrow \infty}} \sum_{t=1}^{T} b_t < \infty, \tag{2.12}$$

$$\underline{\lim_{t \uparrow \infty}} V_{t,2}(x_t^*, x_{t+1}^*; -x_{t+1}^*) \leq 0, \tag{2.13}$$

$$\sum_{t=1}^{\infty} b_t \textit{ exists in } [-\infty, \infty), \tag{2.14}$$

$$\overline{\lim_{t \uparrow \infty}} V_{t,2}(x_t^*, x_{t+1}^*; -x_{t+1}^*) \leq 0. \tag{2.15}$$

Theorem 2.2 is a discrete-time version of Kamihigashi (2001a, Theorem 3.3); recall footnote 6. Conclusion (ii) generalizes the necessity part of Weitzman (1973) except that TVC (2.15) is expressed in terms of generalized directional derivatives instead of support prices. Note that (2.11) and (2.14) follow from Weitzman's assumptions that the return functions are concave and that the objective function is finite for all feasible paths.

Conclusion (ii) also generalizes Ekeland and Scheinkman (1986, Proposition 5.1). Their result assumes the existence of a nonnegative sequence $\{f_t\}$ with $\sum_{t=0}^{\infty} f_t < \infty$ such that for *all* feasible paths $\{x_t\}, \forall t \in \mathbb{Z}_+, V_t(x_t, x_{t+1}) \leq f_t$. Theorem 2.2 does not require this restrictive assumption. Though Ekeland and Scheinkman do not directly assume (2.11) and (2.14), it is shown in Kamihigashi (2000) that their assumptions imply (2.11) and (2.14). See Kamihigashi (2000) for further discussions on Ekeland and Scheinkman's result and approach.

Our last result in this section uses the following assumption.

Assumption 2.5. $\exists \overline{\mu} > 1, \forall \mu \in (1, \overline{\mu}]$, (i) $(x_0^*, \mu x_1^*) \in X_0$, (ii) $\forall t \in \mathbb{N}, (\mu x_t^*, \mu x_{t+1}^*) \in X_t$, (iii) $V_0(x_0^*, \mu x_1^*) > -\infty$, and (iv) $\forall t \in \mathbb{N}, V_t(\mu x_t^*, \mu x_{t+1}^*) \geq V_t(x_t^*, x_{t+1}^*)$.

This assumption means that the optimal path can be shifted proportionally upward ((i) and (ii)) and that such a shift (if sufficiently small) entails a finite loss in period 0 and nonnegative gains in subsequent periods ((iii) and (iv)). The assumption is innocuous at least for standard models with homogeneous returns.

Theorem 2.3. *Assume Assumptions 2.1–2.5. Suppose*

$$\exists \lambda \in [\underline{\lambda}_0, 1), \exists \mu \in (1, \overline{\mu}], \exists \theta \geq 0, \forall t \in \mathbb{N}, \quad \hat{w}_t(\lambda) \leq \theta w_t(\mu). \tag{2.16}$$

Then TVC (2.15) holds.

Theorem 2.3 is similar to Kamihigashi (2001a, Theorem 3.4), but the proofs of these results are quite different. Basically Kamihigashi (Theorem 3.4) uses $\lim_{\mu \downarrow 1} w_t(\mu)$ in (2.16) instead of $w_t(\mu)$ and its proof relies heavily on differentiability and the Euler equation. The proof of Theorem 2.3, on the other hand, verifies (2.11) and (2.14) using (2.16) and Assumption 2.5. Theorem 2.3 is useful particularly in the case of homogenous returns. See Section 4 for applications of the stochastic version of Theorem 2.3.

3 The Stochastic Case

This section extends the results in the preceding section to the stochastic case. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let E denote the associated expectation operator; i.e., $Ez =$

$\int z(\omega)dP(\omega)$ for any random variable $z : \Omega \to \overline{\mathbb{R}}$. When it is important to make explicit the dependence of z on ω, we write $Ez(\omega)$ instead of Ez. Consider the following problem.

$$\text{(3.1)} \qquad \begin{cases} \text{“}\max\limits_{\{x_t\}_{t=0}^{\infty}} & \sum\limits_{t=0}^{\infty} Ev_t(x_t(\omega), x_{t+1}(\omega), \omega)\text{”} \\ \quad \text{s.t.} & x_0 = \overline{x}_0, \quad \forall t \in \mathbb{Z}_+, (x_t, x_{t+1}) \in X_t. \end{cases}$$

For any two sets Y and Z, let $F(Y, Z)$ denote the set of all functions from Y to Z. The following assumption means that x_t is a random variable in a real vector space.

Assumption 3.1. There exists a sequence of real vector spaces $\{B_t\}_{t=0}^{\infty}$ such that $x_0 \in F(\Omega, B_0)$ and $\forall t \in \mathbb{Z}_+, X_t \subset F(\Omega, B_t) \times F(\Omega, B_{t+1})$.

The following assumption simply means that the expression $Ev_t(x_t(\omega), x_{t+1}(\omega), \omega)$ makes sense.

Assumption 3.2. $\forall t \in \mathbb{Z}_+, \forall (y, z) \in X_t$, (i) $\forall \omega \in \Omega, v_t(y(\omega), z(\omega), \omega) \in [-\infty, \infty)$, (ii) the mapping $v_t(y(\cdot), z(\cdot), \cdot) : \Omega \to [-\infty, \infty)$ is measurable, and (iii) $Ev_t(y(\omega), z(\omega), \omega)$ exists in $[-\infty, \infty)$.

We say that a sequence $\{x_t\}$ is a *feasible path* if $x_0 = \overline{x}_0$ and $\forall t \in \mathbb{Z}_+, (x_t, x_{t+1}) \in X_t$. We say that a feasible path $\{x_t^*\}$ is *optimal* if for any feasible path $\{x_t\}$,

$$\text{(3.2)} \qquad \varliminf_{T \uparrow \infty} \sum_{t=0}^{T} [Ev_t(x_t(\omega), x_{t+1}(\omega), \omega) - Ev_t(x_t^*(\omega), x_{t+1}^*(\omega), \omega)] \leq 0.$$

In stochastic optimization problems, feasible paths are usually required to be adapted to a filtration (i.e., x_{t+1} can depend only on information available in period t). Though such an information structure could be imposed here, it is already covered by Assumptions 3.1 and 3.2 as a special case. To be more specific, let $\{\mathcal{F}_t\}$ be a filtration; e.g., $\mathcal{F}_t$ may be the σ-field generated by all possible histories of shocks up to period t. Note that since $\mathcal{F}_t$-measurability implies measurability (more specifically, $\mathcal{F}$-measurability), Assumption 3.2 allows $v_t(y(\cdot), z(\cdot), \cdot)$ to be $\mathcal{F}_t$-measurable. Let M_t be the set of $\mathcal{F}_{t-1}$-measurable functions from Ω to B_t (assuming measurability is well-defined for $F(\Omega, B_t)$.) Then one can require feasible paths to be adapted to $\{\mathcal{F}_t\}$ by assuming $X_t \subset M_t \times M_{t+1}$, i.e., $x_{t+1}(\cdot)$ is $\mathcal{F}_t$-measurable. But this is only a special case of Assumption 3.1 since obviously $M_t \subset F(\Omega, B_t)$. Likewise, other information structures like this are covered by Assumptions 3.1 and 3.2.

At this point, the results in Section 2 can be applied to the present model by defining $V_t : X_t \to [-\infty, \infty)$ and D_t for $t \in \mathbb{Z}_+$ as follows.

$$V_t(x_t, x_{t+1}) = Ev_t(x_t(\omega), x_{t+1}(\omega), \omega), \tag{3.3}$$

$$D_t = F(\Omega, B_t). \tag{3.4}$$

Note that Assumptions 3.1 and 3.2 imply Assumptions 2.1 and 2.2. Hence the model here can be viewed as a deterministic problem. In what follows, we establish stochastic versions of the results in Section 2 with TVCs expressed in terms of v_t instead of V_t.

Like Section 2, this section assumes the existence of an optimal path $\{x_t^*\}$. For simplicity, for $(x_t, x_{t+1}) \in X_t, v_t(x_t, x_{t+1})$ denotes the random variable $v_t(x_t(\cdot), x_{t+1}(\cdot), \cdot) : \Omega \to [-\infty, \infty)$. For $t \in \mathbb{Z}_+$ and $d \in F(\Omega, B_{t+1})$ such that $(x_t^*, x_{t+1}^* + \epsilon d) \in X_t$ for $\epsilon > 0$ sufficiently small, we define the random variable $v_{t,2}(x_t^*, x_{t+1}^*; d)$ as in (2.4); to be precise,

$$v_{t,2}(x_t^*, x_{t+1}^*; d) = \varliminf_{\epsilon \downarrow 0} \frac{v_t(x_t^*, x_{t+1}^* + \epsilon d) - v_t(x_t^*, x_{t+1}^*)}{\epsilon}, \tag{3.5}$$

where $\varliminf_{\epsilon \downarrow 0}$ is applied pointwise (i.e., for each $\omega \in \Omega$).

Remark 3.1. Theorems 3.1–3.3 below hold even if $\varlimsup$ replaces $\varliminf$ in (3.5).[11]

Theorem 3.1. *Assume Assumptions 2.3, 3.1, and 3.2. Suppose* $\forall t \in \mathbb{Z}_+, X_t$ *is convex and* $\forall \omega \in \Omega, v_t(\cdot, \cdot, \omega)$ *is concave. Then*

$$\varliminf_{t \uparrow \infty} Ev_{t,2}(x_t^*, x_{t+1}^*; x_{t+1} - x_{t+1}^*) \leq 0 \tag{3.6}$$

for any feasible path $\{x_t\}$ *satisfying (2.7) and the following:*

$$\forall t \in \mathbb{Z}_+, \exists \epsilon > 0, \quad \zeta_t(\epsilon) \equiv (x_t^*, x_{t+1}^* + \epsilon(x_{t+1} - x_{t+1}^*)) \in X_t,\ Ev_t(\zeta_t(\epsilon)) > -\infty. \tag{3.7}$$

The proof of Theorem 3.1 uses the last inequality in (3.7) in conjunction with the monotone convergence theorem to show

$$\forall t \in \mathbb{Z}_+, \quad Ev_{t,2}(x_t^*, x_{t+1}^*; x_{t+1} - x_{t+1}^*) = V_{t,2}(x_t^*, x_{t+1}^*; x_{t+1} - x_{t+1}^*). \tag{3.8}$$

Given this result, Theorem 3.1 follows immediately from Theorem 2.1.

The rest of this section assumes Assumption 2.4. Stochastic versions of Theorems 2.2 and 2.3 can easily be shown under the following assumption.

[11] See footnote 20 for why this remark is true. See (3.9) for why we use $\varliminf$ in (3.5).

Assumption 3.3. $\forall t \in \mathbb{Z}_+, E v_{t,2}(x_t^*, x_{t+1}^*; -x_{t+1}^*) \leq V_{t,2}(x_t^*, x_{t+1}^*; -x_{t+1}^*)$.

The above inequality can be expressed as

$$(3.9) \qquad E \varliminf_{\epsilon \downarrow 0} \frac{v_t(x_t^*, x_{t+1}^* - \epsilon x_{t+1}^*) - v_t(x_t^*, x_{t+1}^*)}{\epsilon} \leq \varliminf_{\epsilon \downarrow 0} E \frac{v_t(x_t^*, x_{t+1}^* - \epsilon x_{t+1}^*) - v_t(x_t^*, x_{t+1}^*)}{\epsilon}.$$

Of course this holds by Fatou's lemma under the hypothesis of the lemma.[12] This is why we use $\varliminf$ instead of $\varlimsup$ in (2.4) and (3.5).

Remark 3.2. Assumption 3.3 holds (with equality) if $\forall t \in \mathbb{Z}_+, v_t(x_t^*, \lambda x_{t+1}^*)$ is concave in $\lambda \in [\underline{\lambda}_t, 1]$[13] and if $\exists \lambda \in [\underline{\lambda}_t, 1), E v_t(x_t^*, \lambda x_{t+1}^*) > -\infty$, where $\underline{\lambda}_t$ is given by Assumption 2.4.

Remark 3.3. Assumption 3.3 holds by Fatou's lemma if $\forall t \in \mathbb{Z}_+, v_t(x_t^*, \lambda x_{t+1}^*)$ is nonincreasing in $\lambda \in [\underline{\lambda}_t, 1]$ (which is the case in most economic models).

Theorem 3.2. *Assume Assumptions 2.3, 2.4, 3.1–3.3, and (2.11). Then* (i) *(2.12) ⇒ (3.10) and* (ii) *(2.14) ⇒ (3.11), where (3.10) and (3.11) are given by*

$$(3.10) \qquad \varliminf_{t \uparrow \infty} E v_{t,2}(x_t^*, x_{t+1}^*; -x_{t+1}^*) \leq 0,$$

$$(3.11) \qquad \varlimsup_{t \uparrow \infty} E v_{t,2}(x_t^*, x_{t+1}^*; -x_{t+1}^*) \leq 0.$$

Theorem 3.2 is a stochastic version of Theorem 2.2. Conclusion (ii) generalizes the TVC results of Zilcha (1976) and Takekuma (1992) except that our result uses generalized directional derivatives instead of support prices. Our last general result is a stochastic version of Theorem 2.3.

Theorem 3.3. *Under Assumptions 2.3–2.5, 3.1–3.3, and (2.16), TVC (3.11) holds.*

[12]More specifically, (3.9) holds by Fatou's lemma if there exist $\tilde{\epsilon} > 0$ and a measurable function $\xi_t : \Omega \to \mathbb{R}$ such that

$$E\xi_t > -\infty, \qquad \forall \epsilon \in (0, \tilde{\epsilon}), \quad \frac{v_t(x_t^*, x_{t+1}^* - \epsilon x_{t+1}^*) - v_t(x_t^*, x_{t+1}^*)}{\epsilon} \geq \xi_t.$$

[13]To be precise, by "$v_t(x_t^*, \lambda x_{t+1}^*)$ is concave in λ," we mean that with probability one, $v_t(x_t^*, \lambda x_{t+1}^*)$ is a concave function of λ. Likewise any condition involving random variables is understood to hold with probability one.

4 Applications

This section continues to consider the stochastic model (3.1) to offer applications of Theorems 3.2 and 3.3. Assumptions 2.3, 2.4, 3.1, and 3.2 are assumed throughout.

4.1 Homogeneous Returns

In many economic models, the return functions are assumed to be homogenous (e.g., Lucas 1988; Rebelo 1991). In stochastic versions of such models, the following assumption holds.

Assumption 4.1. $\exists \alpha \in \mathbb{R} \setminus \{0\}, \forall t \in \mathbb{N}$, for any $\lambda > 0$ such that $(\lambda x_t^*, \lambda x_{t+1}^*) \in X_t$, we have $v_t(\lambda x_t^*, \lambda x_{t+1}^*) = \lambda^\alpha v_t(x_t^*, x_{t+1}^*)$.

(Proposition 4.3 below deals with the case $\alpha = 0$.) Under certain growth conditions, Alvarez and Stokey (1998) shows the basic results of dynamic programming and the necessity of the STVC for deterministic stationary problems with homogeneous return functions.[14] Here we show that with homogeneous returns, the necessity of the STVC can easily be verified without growth conditions. We use the following standard assumptions.

Assumption 4.2. $\forall t \in \mathbb{Z}_+, Ev_t(\lambda x_t^*, \lambda x_{t+1}^*)$ is nondecreasing in $\lambda \in [\underline{\lambda}_0, 1]$, where $\underline{\lambda}_0$ is given by Assumption 2.4.

Assumption 4.3. $\forall t \in \mathbb{Z}_+, v_t(x_t^*, \lambda x_{t+1}^*)$ is nonincreasing in $\lambda \in [\underline{\lambda}_t, 1]$.

Remark 4.1. Assumption 4.3 and TVC (3.11) imply

(4.1)
$$\lim_{t \uparrow \infty} Ev_{t,2}(x_t^*, x_{t+1}^*; -x_{t+1}^*) = 0,$$

which is a stochastic version of the STVC.

Proposition 4.1. *Assume Assumptions 2.3, 2.4, 3.1, 3.2, and 4.1–4.3. Suppose*

(4.2)
$$-\infty < \sum_{t=1}^{\infty} Ev_t(x_t^*, x_{t+1}^*) < \infty.$$

Then TVC (4.1) holds.[15]

[14] Boyd's (1990) symmetry approach, which does not directly deal with TVCs, is also useful in analyzing problems with homogeneous returns.

[15] The infinite sum in (4.2) exists since $\forall t \in \mathbb{N}, \alpha Ev_t(x_t^*, x_{t+1}^*) \geq 0$. See (A.30).

The proof of Proposition 4.1 uses Theorem 3.2. Condition (4.2) is usually assumed or taken for granted in applied studies. Proposition 4.1 shows that the STVC is necessary in such cases. Condition (4.2), however, is unnecessary under Assumption 2.5, which allows us to apply Theorem 3.3.

Proposition 4.2. *Under Assumptions 2.3–2.5, 3.1, 3.2, and 4.1–4.3, TVC (4.1) holds.*

Propositions 4.1 and 4.2 indicate that for models with homogeneous returns, there is essentially no issue about necessity of the STVC.

4.2 A Stochastic Growth Model

Consider the following problem.

$$\begin{cases} \text{“}\max\limits_{\{x_t\}_{t=0}^{\infty}} & \sum\limits_{t=0}^{\infty} E\beta^t u(g_t(x_t(\omega), x_{t+1}(\omega), \omega))\text{”} \\ \text{s.t.} & x_0 = \overline{x}_0, \quad \forall t \in \mathbb{Z}_+, (x_t, x_{t+1}) \in X_t. \end{cases} \tag{4.3}$$

As in Section 3, $g_t(x_t, x_{t+1})$ denotes the random variable $g_t(x_t(\cdot), x_{t+1}(\cdot), \cdot)$. Various stochastic growth models (e.g., Brock and Mirman (1972)) take the form of (4.3). One may call x_t the capital stock (or the vector of capital stocks) at the beginning of period t and $g_t(x_t, x_{t+1})$ consumption in period t.[16] For simplicity, we assume the following.

Assumption 4.4. (i) $\beta \in (0, 1)$, (ii) $u : \mathbb{R}_+ \to [-\infty, \infty)$, and (iii) $\exists \alpha \in (-\infty, 1]$,

$$u(\cdot) = \begin{cases} \dfrac{(\cdot)^\alpha}{\alpha} & \text{if } \alpha \neq 0, \\ \ln(\cdot) & \text{if } \alpha = 0. \end{cases} \tag{4.4}$$

Assumption 4.5. $\forall t \in \mathbb{Z}_+$, (i) X_t is convex, (ii) $(x_t^*, 0), (0, 0) \in X_t$, (iii) $g_t(0, 0) \geq 0$, (iv) $g_t(x_t^*, x_{t+1}^*) > 0$, and (v) $\forall \omega \in \Omega, g_t(\cdot, \cdot, \omega)$ is concave.

Assumption 4.6. $\forall t \in \mathbb{Z}_+, g_t(x_t^*, \lambda x_{t+1}^*)$ is nonincreasing and continuous in $\lambda \in (0, 1]$.

Remark 4.2. Assumptions 4.4–4.6 and TVC (3.11) imply

$$\lim_{t \uparrow \infty} \beta^t E[u'(g_t(x_t^*, x_{t+1}^*)) g_{t,2}(x_t^*, x_{t+1}^*; -x_{t+1}^*)] = 0, \tag{4.5}$$

[16] One can apply the results in this section to models with endogenous labor supply such as RBC models (e.g., King, Rebelo, and Plosser, 1988). To do so, one may take the optimal labor path as given and consider the maximization problem over consumption and capital paths.

where $g_{t,2}(x_t^*, x_{t+1}^*; -x_{t+1}^*)$ is defined as in (3.5).[17]

Proposition 4.3. *Assume Assumptions 2.3, 3.1, 3.2, and 4.4–4.6. Suppose* $\alpha = 0$. *Then TVC (4.5) holds.*

Proposition 4.4. *Assume Assumptions 2.3, 3.1, 3.2, and 4.4–4.6. Suppose* $\alpha \neq 0$ *and*

$$-\infty < \sum_{t=1}^{\infty} \beta^t E u(g_t(x_t^*, x_{t+1}^*)) < \infty. \tag{4.6}$$

Then TVC (4.5) holds.[18]

The proofs of Propositions 4.3 and 4.4 use Theorem 3.2. Proposition 4.3 shows that the STVC is always necessary in the logarithmic case. Even in the non-logarithmic case, Proposition 4.4 shows that the STVC is guaranteed to be necessary unless one is willing to allow lifetime utility to be infinite at the optimum. Such cases are rare in practice since (4.6) is usually assumed or taken for granted in applied studies.

Even without (4.6), however, Theorem 3.3 can be invoked under Assumption 2.5. Indeed, if $\forall t \in \mathbb{N}, g_t$ is homogenous, then Proposition 4.2 (a consequence of Theorem 3.3) directly applies. We also have a useful result that does not assume homogeneity. The result, whose statement is slightly complicated, is stated in Appendix B.

5 Concluding Remarks

In this paper we showed (i) an extension of Michel's (1990, Theorem 1) necessity result to our abstract reduced-form model, (ii) a generalization of the TVC results of Weitzman (1973) and Ekeland and Scheinkman (1986), and (iii) a new result that is useful particularly in the case of homogeneous returns. These results were shown for an extremely general deterministic reduced-form model that does not assume differentiability, continuity, or concavity, and that imposes virtually no restriction on the state spaces. The three results were further extended to a stochastic reduced-form model. The stochastic extensions were easily accomplished

[17] This remark can easily be verified by showing $v_{t,2}(x_t^*, x_{t+1}^*; -x_{t+1}^*) = u'(g_t(x_t^*, x_{t+1}^*))g_{t,2}(x_t^*, x_{t+1}^*; -x_{t+1}^*)$ using the mean value theorem. This is why the continuity requirement in Assumption 4.6 is needed.

[18] Proposition 4.4 can easily be generalized to more general utility functions. See footnote 22 for details.

since our deterministic model is so general that the stochastic model is in fact a special case of the deterministic model.

As examples of applications, we studied two special cases of the stochastic model. The first was a stochastic reduced-form model with homogeneous returns. For that model, we showed that the STVC is necessary under standard assumptions, even when the objective function is not guaranteed to be finite at the optimum. The second special case was a general type of stochastic growth model with CRRA utility. For that model, we showed that the STVC is necessary if utility is logarithmic or if the objective function is finite at the optimum.[19]

As those special cases illustrate, our general stochastic results are highly useful. They are significant as well not only because the existing literature does not provide widely applicable results on necessity of TVCs for stochastic problems, but also because our stochastic results were established at the same level of generality as that of our very general deterministic results. The results of this paper suggest that as far as necessity of TVCs is concerned, there is little difference between deterministic and stochastic cases.

A Proofs

This appendix proves the results stated in the main text. Section A.1 considers a one-dimensional version of the reduced-form model studied in Section 2. Sections A.2–A.11 prove the results stated in the main text. All the theorems in Section 2 are derived from the results in Section A.1.

A.1 General Results

Consider the following problem.

$$(\text{A.1}) \qquad \begin{cases} \text{“} \max\limits_{\{y_t\}_{t=0}^{\infty}} & \sum\limits_{t=0}^{\infty} r_t(y_t, y_{t+1}) \text{”} \\ \quad \text{s.t.} & y_0 = \overline{y}_0, \ \ \forall t \in \mathbb{Z}_+, (y_t, y_{t+1}) \in Y_t. \end{cases}$$

Assumption A.1. $\overline{y}_0 \in \mathbb{R}$ and $\forall t \in \mathbb{Z}_+, Y_t \subset \mathbb{R} \times \mathbb{R}$.

[19] Appendix B shows a result that does not require the finiteness of the objective function.

Assumption A.2. $r_t : Y_t \to [-\infty, \infty)$.

Assumption A.1 says that y_t is one-dimensional. Feasible paths and optimal paths are defined as in Section 2.

Assumption A.3. There exists an optimal path $\{y_t^*\}$.

Assumption A.4. $\forall t \in \mathbb{Z}_+, \exists \bar{\epsilon}_t > 0, \forall \epsilon \in (0, \bar{\epsilon}_t], (y_t^*, y_{t+1}^* - \epsilon) \in Y_t$ and $\forall \tau \geq t+1, (y_\tau^* - \epsilon, y_{\tau+1}^* - \epsilon) \in Y_\tau$.

Assumption A.4 means that the optimal path can be shifted uniformly downward starting from any period. For $t \in \mathbb{Z}_+$ and $d \in \mathbb{R}$, define $r_{t,2}(y_t^*, y_{t+1}^*; d)$ as in (2.4).

Remark A.1. Theorems A.1 and A.2 below hold even if $\overline{\lim}$ replaces $\underline{\lim}$ in (2.4).[20]

For $t \in \mathbb{Z}_+$, define

$$q_t = r_{t,2}(y_t^*, y_{t+1}^*; -1). \tag{A.2}$$

For $t \in \mathbb{N}$ and $\epsilon \in \mathbb{R} \setminus \{0\}$ with $(y_t^* - \epsilon, y_{t+1}^* - \epsilon) \in Y_t$, define

$$m_t(\epsilon) = \frac{r_t(y_t^*, y_{t+1}^*) - r_t(y_t^* - \epsilon, y_{t+1}^* - \epsilon)}{\epsilon}, \tag{A.3}$$

$$\hat{m}_t(\epsilon) = \sup_{\tilde{\epsilon} \in (0, \epsilon]} m_t(\tilde{\epsilon}), \tag{A.4}$$

where $\hat{m}_t(\epsilon)$ is defined only for $\epsilon \in (0, \bar{\epsilon}_0]$ ($\bar{\epsilon}_0$ is given by Assumption A.4).

Lemma A.1. *Under Assumptions A.1–A.4,*

$$\forall s \in \mathbb{Z}_+, \quad q_s \leq \underline{\lim}_{\epsilon \downarrow 0} \overline{\lim}_{T \uparrow \infty} \sum_{t=s+1}^{T} m_t(\epsilon). \tag{A.5}$$

Proof. Let $s \in \mathbb{Z}_+$ and $\epsilon \in (0, \bar{\epsilon}_s]$. By the optimality of $\{y_t^*\}$ and Assumption A.4,

$$\underline{\lim}_{T \uparrow \infty} \left[r_s(y_s^*, y_{s+1}^* - \epsilon) - r_s(y_s^*, y_{s+1}^*) + \sum_{t=s+1}^{T} [r_t(y_t^* - \epsilon, y_{t+1}^* - \epsilon) - r_t(y_t^*, y_{t+1}^*)] \right] \leq 0. \tag{A.6}$$

Dividing through by ϵ and rearranging, we get

$$\frac{r_s(y_s^*, y_{s+1}^* - \epsilon) - r_s(y_s^*, y_{s+1}^*)}{\epsilon} \leq \overline{\lim}_{T \uparrow \infty} \sum_{t=s+1}^{T} \frac{r_t(y_t^*, y_{t+1}^*) - r_t(y_t^* - \epsilon, y_{t+1}^* - \epsilon)}{\epsilon}. \tag{A.7}$$

Recalling (A.3) and applying $\underline{\lim}_{\epsilon \downarrow 0}$ yields (A.5). □

[20] This can be seen by replacing $\underline{\lim}$ with $\overline{\lim}$ in the last sentence of the proof of Lemma A.1 and (A.5). Lemma A.1 is the only place where (2.4) is used.

Define

$$\Psi = \{\{f_t\}_{t=1}^{\infty} \subset \mathbb{R} \,|\, \overline{\lim}_{T\uparrow\infty} \textstyle\sum_{t=1}^{T} f_t \in [-\infty, \infty)\}, \tag{A.8}$$

$$\Phi = \{\{f_t\}_{t=1}^{\infty} \subset \mathbb{R} \,|\, \lim_{T\uparrow\infty} \textstyle\sum_{t=1}^{T} f_t \text{ exits in } [-\infty, \infty)\}. \tag{A.9}$$

Lemma A.2. (i) $\forall \{f_t\} \in \Psi, \underline{\lim}_{s\uparrow\infty} \overline{\lim}_{T\uparrow\infty} \sum_{t=s}^{T} f_t \leq 0.$[21]
(ii) $\forall \{f_t\} \in \Phi, \overline{\lim}_{s\uparrow\infty} \overline{\lim}_{T\uparrow\infty} \sum_{t=s}^{T} f_t \leq 0.$

Proof. Let $\{f_t\} \in \Psi$ and $A = \overline{\lim}_{T\uparrow\infty} \sum_{t=1}^{T} f_t$ $(< \infty)$. Then

$$\forall s \geq 2, \qquad \overline{\lim_{T\uparrow\infty}} \sum_{t=s}^{T} f_t = A - \sum_{t=1}^{s-1} f_t. \tag{A.10}$$

Thus if $A = -\infty$, then $\underline{\lim}_{s\uparrow\infty} \overline{\lim}_{T\uparrow\infty} \sum_{t=s}^{T} f_t = -\infty$. If $A > -\infty$, then

$$\underline{\lim_{s\uparrow\infty}} \overline{\lim_{T\uparrow\infty}} \sum_{t=s}^{T} f_t = A - \overline{\lim_{s\uparrow\infty}} \sum_{t=1}^{s-1} f_t = A - A = 0. \tag{A.11}$$

Hence (i) holds. The proof of (ii) is similar. □

Theorem A.1. *Assume Assumptions A.1–A.4. Suppose*

$$\exists \{b_t\}_{t=1}^{\infty} \subset \mathbb{R}, \exists \epsilon \in (0, \overline{\epsilon}_0], \forall t \in \mathbb{N}, \qquad \hat{m}_t(\epsilon) \leq b_t. \tag{A.12}$$

Then (i) $\{b_t\} \in \Psi \Rightarrow \underline{\lim}_{t\uparrow\infty} q_t \leq 0$ *and* (ii) $\{b_t\} \in \Phi \Rightarrow \overline{\lim}_{t\uparrow\infty} q_t \leq 0$.

Proof. By (A.5) and (A.12), $\forall s \in \mathbb{Z}_+, q_s \leq \overline{\lim}_{T\uparrow\infty} \sum_{t=s+1}^{T} b_t$. Thus both (i) and (ii) hold by Lemma A.2. □

Assumption A.5. $\exists \overline{\delta} > 0, \forall \delta \in (0, \overline{\delta}]$, (i) $(y_0^*, y_1^* + \delta) \in Y_0$, (ii) $\forall t \in \mathbb{N}, (y_t^* + \delta, y_{t+1}^* + \delta) \in Y_t$, (iii) $r_0(y_0^*, y_1^* + \delta) > -\infty$, and (iv) $\forall t \in \mathbb{N}, r_t(y_t^* + \delta, y_{t+1}^* + \delta) \geq r_t(y_t^*, y_{t+1}^*)$.

Theorem A.2. *Assume Assumptions A.1–A.5. Suppose*

$$\exists \epsilon \in (0, \overline{\epsilon}_0], \exists \delta \in (0, \overline{\delta}], \exists \theta \geq 0, \forall t \in \mathbb{N}, \qquad \hat{m}_t(\epsilon) \leq \theta m_t(-\delta). \tag{A.13}$$

Then $\overline{\lim}_{t\uparrow\infty} q_t \leq 0$.

[21] A similar result is shown by Michel (1990, Proposition 1). Continuous-time versions of Lemma A.2 and some of the other results in this paper are shown in Kamihigashi (2001a).

Proof. By the optimality of $\{y_t^*\}$ and Assumption A.5,

$$r_0(y_0^*, y_1^* + \delta) - r_0(y_0^*, y_1^*) + \sum_{t=1}^{\infty}[r_t(y_t^* + \delta, y_{t+1}^* + \delta) - r_t(y_t^*, y_{t+1}^*)] \leq 0, \tag{A.14}$$

where the infinite sum exists by Assumption A.5(iv). By Assumption A.5(iii), for (A.14) to hold, we must have

$$\sum_{t=1}^{\infty}[r_t(y_t^* + \delta, y_{t+1}^* + \delta) - r_t(y_t^*, y_{t+1}^*)] < \infty. \tag{A.15}$$

Dividing through by δ and recalling (A.3), we get $\sum_{t=1}^{\infty} m_t(-\delta) < \infty$. Thus by (A.13) and Theorem A.1(ii), $\overline{\lim}_{t\uparrow\infty} q_t \leq 0$. □

A.2 Proof of Theorem 2.1

Let $\{x_t\}$ be a feasible path satisfying (2.7) and (2.8). Then

$$\overline{\lim_{T\uparrow\infty}} \sum_{t=0}^{T}[V_t(x_t^*, x_{t+1}^*) - V_t(x_t, x_{t+1})] < \infty. \tag{A.16}$$

It follows that

$$\{V_t(x_t^*, x_{t+1}^*) - V_t(x_t, x_{t+1})\}_{t=1}^{\infty} \in \Psi, \tag{A.17}$$

where Ψ is defined by (A.8). For $t \in \mathbb{Z}_+$, let $e_t = x_t^* - x_t$. Consider the following problem.

$$\begin{cases} \text{“}\max\limits_{\{y_t\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} V_t(x_t^* + y_t e_t, x_{t+1}^* + y_{t+1}e_{t+1})\text{”} \\ \quad\text{s.t.} & y_0 = 0, \quad \forall t \in \mathbb{Z}_+, (x_t^* + y_t e_t, x_{t+1}^* + y_{t+1}e_{t+1}) \in X_t, \end{cases} \tag{A.18}$$

where $\forall t \in \mathbb{Z}_+, y_t \in \mathbb{R}$. For $t \in \mathbb{Z}_+$, define

$$r_t(y_t, y_{t+1}) = V_t(x_t^* + y_t e_t, x_{t+1}^* + y_{t+1}e_{t+1}), \tag{A.19}$$

$$Y_t = \{(y_t, y_{t+1}) \in \mathbb{R}^2 \,|\, (x_t^* + y_t e_t, x_{t+1}^* + y_{t+1}e_{t+1}) \in X_t\}. \tag{A.20}$$

Obviously Assumptions A.1 and A.2 hold. For $t \in \mathbb{Z}_+$, let $y_t^* = 0$. Then $\{y_t^*\}$ is optimal for (A.18) since $\{x_t^*\}$ is optimal for (2.1). Thus Assumption A.3 holds. Assumption A.4 follows from the convexity of X_t and (2.8). Note that $\forall t \in \mathbb{Z}_+$,

$$q_t = \lim_{\epsilon\downarrow 0} \frac{V_t(x_t^*, x_{t+1}^* - \epsilon e_t) - V_t(x_t^*, x_{t+1}^*)}{\epsilon} = V_{t,2}(x_t^*, x_{t+1}^*; -e_t). \tag{A.21}$$

Note also that $\forall t \in \mathbb{Z}_+, \forall \epsilon \in (0,1]$,

$$m_t(\epsilon) = \frac{V_t(x_t^*, x_{t+1}^*) - V_t(x_t^* - \epsilon e_t, x_{t+1}^* - \epsilon e_{t+1})}{\epsilon} \tag{A.22}$$

$$\leq V_t(x_t^*, x_{t+1}^*) - V_t(x_t^* - e_t, x_{t+1}^* - e_{t+1}), \tag{A.23}$$

where the inequality holds by concavity. Recalling (A.17) and the definition of $\{e_t\}$, we see that (A.12) holds with $b_t = V_t(x_t^*, x_{t+1}^*) - V_t(x_t, x_{t+1})$. Thus TVC (2.6) holds by Theorem A.1(i) and (A.21).

A.3 Proof of Theorem 2.2

Consider the following problem.

$$\begin{cases} \text{“}\max\limits_{\{y_t\}_{t=0}^{\infty}} & \sum\limits_{t=0}^{\infty} V_t(y_t x_t^*, y_{t+1} x_{t+1}^*)\text{”} \\ \quad\text{s.t.} & y_0 = 1, \quad \forall t \in \mathbb{Z}_+, (y_t x_t^*, y_{t+1} x_{t+1}^*) \in X_t, \end{cases} \tag{A.24}$$

where $\forall t \in \mathbb{Z}_+, y_t \in \mathbb{R}$. For $t \in \mathbb{Z}_+$, define

$$r_t(y_t, y_{t+1}) = V_t(y_t x_t^*, y_{t+1} x_{t+1}^*), \tag{A.25}$$

$$Y_t = \{(y_t, y_{t+1}) \in \mathbb{R}^2 \,|\, (y_t x_t^*, y_{t+1} x_{t+1}^*) \in X_t\}. \tag{A.26}$$

Obviously Assumptions A.1 and A.2 hold. For $t \in \mathbb{Z}_+$, let $y_t^* = 1$. Then $\{y_t^*\}$ is optimal for (A.24) since $\{x_t^*\}$ is optimal for (2.1). Thus Assumption A.3 holds. Assumption A.4 follows from Assumption 2.4. Note that $\forall t \in \mathbb{Z}_+$,

$$q_t = \lim_{\epsilon \downarrow 0} \frac{V_t(x_t^*, (1-\epsilon) x_{t+1}^*) - V_t(x_t^*, x_{t+1}^*)}{\epsilon} = V_{t,2}(x_t^*, x_{t+1}^*; -x_{t+1}^*). \tag{A.27}$$

Note also that $\forall t \in \mathbb{N}, \hat{m}_t(1-\lambda) = \hat{w}_t(\lambda)$. Thus (A.12) holds by (2.11). Hence both conclusions hold by Theorem A.1.

A.4 Proof of Theorem 2.3

Consider (A.24) again. For $t \in \mathbb{Z}_+$, define r_t and Y_t by (A.25) and (A.26). It is easy to see that Assumptions A.1–A.5 hold. Note that $\forall t \in \mathbb{N}, \hat{m}_t(1-\lambda) = \hat{w}_t(\lambda)$ and $m_t(1-\mu) = w_t(\mu)$. Thus (A.13) holds by (2.16). Recalling (A.27), we see that TVC (2.15) holds by Theorem A.2.

A.5 Proof of Theorem 3.1

By Theorem 2.1, TVC (2.6) holds. Thus it suffices to verify (3.8). Let $t \in \mathbb{Z}_+$ and $e_t = x_t^* - x_t$. For $\tilde{\epsilon} \in (0, \epsilon]$ (ϵ is given by (3.7)), define

$$a_t(\tilde{\epsilon}) = \frac{v_t(x_t^*, x_{t+1}^* - \tilde{\epsilon} e_{t+1}) - v_t(x_t^*, x_{t+1}^*)}{\tilde{\epsilon}}. \tag{A.28}$$

By (3.7), $E a_t(\epsilon) > -\infty$. By concavity, $a_t(\tilde{\epsilon})$ is nonincreasing in $\tilde{\epsilon} \in (0, \epsilon]$. Hence $\forall \tilde{\epsilon} \in (0, \epsilon]$, $a_t(\tilde{\epsilon}) \geq a_t(\epsilon)$. Now by the monotone convergence theorem, $\lim_{\tilde{\epsilon} \downarrow 0} E a_t(\tilde{\epsilon}) = E \lim_{\tilde{\epsilon} \downarrow 0} a_t(\tilde{\epsilon})$, which is equivalent to (3.8).

A.6 Proof of Theorem 3.2

Immediate from Theorem 2.2 and Assumption 3.3.

A.7 Proof of Theorem 3.3

Immediate from Theorem 2.3 and Assumption 3.3.

A.8 Proof of Proposition 4.1

Note from Remark 3.3 that

$$\text{Assumption 4.3} \Rightarrow \text{Assumption 3.3}. \tag{A.29}$$

Thus to conclude TVC (4.1) from Theorem 3.2 and Remark 4.1, it suffices to verify (2.11) and (2.14). Let $\underline{\lambda} \in (\underline{\lambda}_0, 1)$, where $\underline{\lambda}_0$ is given by Assumption 2.4. Let $t \in \mathbb{N}$. By Assumptions 4.1 and 4.2,

$$0 \leq \left. \frac{\partial E v_t(\lambda x_t^*, \lambda x_{t+1}^*)}{\partial \lambda} \right|_{\lambda=1} = \left. \frac{\partial \lambda^\alpha E v_t^*}{\partial \lambda} \right|_{\lambda=1} = \alpha E v_t^*, \tag{A.30}$$

where $v_t^* = v_t(x_t^*, x_{t+1}^*)$. By Assumption 4.1 and (A.30),

$$\forall \lambda \in [\underline{\lambda}, 1), \qquad w_t(\lambda) = \frac{1 - \lambda^\alpha}{(1 - \lambda)\alpha} \alpha E v_t^* \leq A \alpha E v_t^*, \tag{A.31}$$

where $A = \sup_{\lambda \in [\underline{\lambda}, 1)} (1 - \lambda^\alpha)/[(1 - \lambda)\alpha] \in (0, \infty)$; A is finite since $(1 - \lambda^\alpha)/[(1 - \lambda)\alpha]$ is continuous on $[\underline{\lambda}, 1]$ by l'Hôpital's rule. Now (2.11) and (2.14) follow from (A.31) and (4.2).

A.9 Proof of Proposition 4.2

Recall (A.29). To conclude TVC (4.1) from Theorem 3.3 and Remark 4.1, it suffices to verify (2.16). Let $\mu \in (1, \overline{\mu}]$, where $\overline{\mu}$ is given by Assumption 2.5. By Assumption 4.1,

$$\forall t \in \mathbb{N}, \qquad w_t(\mu) = \frac{\mu^\alpha - 1}{(\mu - 1)\alpha} \alpha E v_t^*. \tag{A.32}$$

Now (2.16) follows from (A.31) and (A.32).

A.10 Proof of Proposition 4.3

Note from Remarks 2.3 and 3.3 that

$$\text{Assumptions 4.5 and 4.6} \Rightarrow \text{Assumption 2.4 and 3.3.} \tag{A.33}$$

To conclude TVC (4.1) from Theorem 3.2 and Remark 4.2, it suffices to verify (2.11) and (2.14). Let $\underline{\lambda} \in (0, 1), \lambda \in [\underline{\lambda}, 1)$, and $t \in \mathbb{N}$. By concavity and Assumption 4.5(iii),

$$g_t(\lambda x_t^*, \lambda x_{t+1}^*) \geq \lambda g_t(x_t^*, x_{t+1}^*) + (1 - \lambda) g_t(0, 0) \geq \lambda g_t(x_t^*, x_{t+1}^*). \tag{A.34}$$

We have

$$(1 - \lambda)\beta^{-t} w_t(\lambda) = E[\ln g_t(x_t^*, x_{t+1}^*) - \ln g_t(\lambda x_t^*, \lambda x_{t+1}^*)] \tag{A.35}$$

$$\leq E[\ln g_t(x_t^*, x_{t+1}^*) - \ln(\lambda g_t(x_t^*, x_{t+1}^*))] = -\ln \lambda, \tag{A.36}$$

where the inequality holds by (A.34). It follows that

$$w_t(\lambda) \leq \beta^t \frac{-\ln \lambda}{1 - \lambda} \leq \beta^t \frac{-\ln \underline{\lambda}}{1 - \underline{\lambda}}. \tag{A.37}$$

Now (2.11) and (2.14) follow.

A.11 Proof of Proposition 4.4

Recall (A.33). To conclude TVC (4.1) from Theorem 3.2 and Remark 4.2, it suffices to verify (2.11) and (2.14). Let $\underline{\lambda} \in (0, 1), \lambda \in [\underline{\lambda}, 1)$, and $t \in \mathbb{N}$. We have

$$(1 - \lambda)\beta^{-t} w_t(\lambda) = E[u(g_t(x_t^*, x_{t+1}^*)) - u(g_t(\lambda x_t^*, \lambda x_{t+1}^*))] \tag{A.38}$$

$$\leq E[u(g_t(x_t^*, x_{t+1}^*)) - u(\lambda g_t(x_t^*, x_{t+1}^*))] \tag{A.39}$$

$$= (1 - \lambda^\alpha) E u(g_t(x_t^*, x_{t+1}^*)), \tag{A.40}$$

where (A.39) holds by (A.34), and (A.40) holds by Assumption 4.4. It follows that

$$w_t(\lambda) \leq \beta^t \frac{1-\lambda^\alpha}{(1-\lambda)\alpha} \alpha E u(g_t^*) \leq \beta^t A \alpha E u(g_t^*), \tag{A.41}$$

where $g_t^* = g_t(x_t^*, x_{t+1}^*)$ and A is as in (A.31). Now (2.11) and (2.14) follow from (4.6) and (A.41).[22]

B A Further Result on the Model of Section 4.2

This appendix considers the model of Section 4.2 and shows a result that does not assume (4.6), i.e., the finiteness of the objective function at the optimum.

Proposition B.1. *Assume Assumptions 2.3, 2.5, 3.1, 3.2, and 4.4–4.6. Suppose* $\alpha \neq 0$. *For* $\lambda \in [0, \overline{\mu}]$, *let* $\gamma_t(\lambda) = g_t(\lambda x_t^*, \lambda x_{t+1}^*)$, *where* $\overline{\mu}$ *is given by Assumption 2.5. Let* $\overline{\epsilon} = \min\{1, \overline{\mu} - 1\}$. *Suppose*

$$\exists \epsilon \in (0, \overline{\epsilon}), \exists \theta \geq 0, \forall t \in \mathbb{N}, \qquad 0 \leq \gamma_t(1) - \gamma_t(1-\epsilon) \leq \theta[\gamma_t(1+\epsilon) - \gamma_t(1)]. \tag{B.1}$$

Then TVC (4.5) holds.[23]

Proof. Recall (A.33). To conclude TVC (4.1) from Theorem 3.3 and Remark 4.2, it suffices to verify (2.16). Let $\lambda = 1 - \epsilon, \mu = 1 + \epsilon$, and $t \in \mathbb{N}$. Note from (A.38) that by concavity,

$$w_t(\lambda) \leq \frac{\beta^t E u'(\gamma_t(\lambda))[\gamma_t(1) - \gamma_t(\lambda)]}{1-\lambda} \leq \frac{\beta^t \lambda^{\alpha-1} E u'(\gamma_t(1))[\gamma_t(1) - \gamma_t(\lambda)]}{1-\lambda}, \tag{B.2}$$

where the second inequality uses (A.34), the first inequality in (B.1), and Assumption 4.4. By concavity, $\gamma_t(1) \geq (1 - (1/\mu))\gamma_t(0) + \gamma_t(\mu)/\mu \geq \gamma_t(\mu)/\mu$ (the second inequality holds by Assumption 4.5(iii)); thus

$$\gamma_t(\mu) \leq \mu \gamma_t(1). \tag{B.3}$$

[22]This proof works without Assumption 4.4 if u is nondecreasing, concave, and differentiable and if $\exists \underline{\lambda} \in (0,1), \exists \alpha \in (-\infty, 1], \forall t \in \mathbb{N}, \forall \lambda \in [\underline{\lambda}, 1), u(\lambda g_t^*) \geq \lambda^\alpha u(g_t^*)$. In this case, "=" in (A.40) must be replaced by "≤." That $\alpha E u(g_t^*) \geq 0$ can be shown from the inequality $u(\lambda g_t^*) \geq \lambda^\alpha u(g_t^*)$. Note that both sides are equal at $\lambda = 1$. Thus, differentiating both sides with respect to λ and evaluating at $\lambda = 1$, we have $0 \leq u'(g_t^*) g_t^* \leq \alpha u(g_t^*)$.

[23]Proposition B.1 can easily be generalized to more general utility functions. In fact the proof of Proposition B.1 works without Assumption 4.4 if u is nondecreasing, concave, and differentiable and if $\exists \sigma \geq 0, \forall t \in \mathbb{N}, u'(\gamma_t(1-\epsilon)) \leq \sigma u'(\gamma_t(1+\epsilon))$, where ϵ is given by (B.1). (To verify this inequality, one may utilize (A.34) and (B.3).) In this case, the first inequalities in (B.2) and (B.4) together with (B.1) imply (2.16).

By concavity,

$$\text{(B.4)} \qquad w_t(\mu) \geq \frac{\beta^t E u'(\gamma_t(\mu))[\gamma_t(\mu) - \gamma_t(1)]}{\mu - 1} \geq \frac{\beta^t \mu^{\alpha-1} E u'(\gamma_t(1))[\gamma_t(\mu) - \gamma_t(1)]}{\mu - 1},$$

where the second inequality uses (B.3) and Assumption 4.4. Recalling Remark 2.4, we obtain (2.16) from (B.1), (B.2), and (B.4). □

Note that (B.1) holds with $\theta = 1$ if $\exists n \in \mathbb{N}, \forall t \in \mathbb{N}$, (i) $B_t = \mathbb{R}^n$ (recall Assumption 3.1), (ii) $g_t(x_t^*, x_{t+1}^*) \geq g_t(0, 0)$, and (iii) $\forall (y, z) \in X_t, g_t(y, z) = a_t + b_t y + c_t z$ for some $a_t : \Omega \to \mathbb{R}$ and $b_t, c_t : \Omega \to \mathbb{R}^n$. These conditions are satisfied in single-agent asset pricing models of the type studied by Lucas (1978), Kamihigashi (1998), and Montrucchio and Privileggi (2000). For such models, TVC (4.5) can be used to rule out bubbles.[24] Even if g_t is nonlinear, condition (B.1) can be satisfied; e.g., it is satisfied in a stochastic discrete-time version of the asset pricing model with nonlinear constraints discussed in Kamihigashi (2001a).

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[24]The TVC by itself does not rule out bubbles since the Euler equation may be violated; see Kamihigashi (1998) and Montrucchio and Privileggi (2000). See Santos and Woodford (1997) and Montrucchio (2000) for results on bubbles in economies with heterogeneous agents.

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